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ENTIRE OR RATIONAL MAPS WITH INTEGER MULTIPLIERS

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ABSTRACT. Let $\mathcal{O}_K$ be the ring of integers of an imaginary quadratic field K. Recently, Ji and Xie proved that every rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$ whose multipliers all lie in $\mathcal{O}_K$ is a power map, a Chebyshev map or a Lattès map. Their proof relies on a result from non-Archimedean dynamics obtained by Rivera-Letelier. In the present note, we show that one can avoid using this result by considering a differential equation instead. Our proof of Ji and Xie's result also applies to the case of entire maps. Thus, we also show that every nonaffine entire map $f\colon \mathbb{C} \to \mathbb{C}$ whose multipliers all lie in $\mathcal{O}_K$ is a power map or a Chebyshev map.

1. Introduction

Suppose that S is a Riemann surface and $f\colon S \to S$ is a holomorphic map. We recall that a point $z_0 \in S$ is *periodic* for f if there exists an integer $p \geq 1$ such that $f^{\circ p}(z_0) = z_0$. In this case, the least such integer p is called the *period* of z_0. The *multiplier* of f at z_0 is the unique eigenvalue $\lambda \in \mathbb{C}$ of the differential of $f^{\circ p}$ at z_0. By the chain rule, the multiplier is invariant under conjugation: if $\phi\colon S \to S$ is a biholomorphism and $g = \phi \circ f \circ \phi^{-1}$, then $\phi(z_0)$ is periodic for g with period p and multiplier λ. In this note, we will assume that S represents either the complex line $\mathbb{C}$ or the Riemann sphere $\widehat{\mathbb{C}}$, and thus the map f will be either entire or rational.

A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$ is said to be

- a *power map* if it is conjugate to $z \mapsto z^{\pm d}$,
- a *Chebyshev map* if it is conjugate to $\pm T_d$, where T_d is the dth Chebyshev polynomial,
- a *Lattès map* if there exist a 1-dimensional complex torus $\mathbb{T}$, a holomorphic map $L\colon \mathbb{T} \to \mathbb{T}$ and a nonconstant holomorphic map $p\colon \mathbb{T} \to \widehat{\mathbb{C}}$ that make the following diagram commute:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\;L\;} & \mathbb{T} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \widehat{\mathbb{C}} & \xrightarrow[\;f\;]{} & \widehat{\mathbb{C}} \end{array}$$

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Power maps, Chebyshev maps and Lattès maps are called finite quotients of affine maps by Milnor in [Mil06] and exceptional maps by Ji and Xie in [JX23]. In this note, we will use the second terminology.

As shown by Milnor in [Mil06], if $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a power map, a Chebyshev map or a Lattès map, then its multipliers at its periodic points all belong to the ring of integers $\mathcal{O}_K$ of some imaginary quadratic field $K \subset \mathbb{C}$. Milnor conjectured that the converse is true. In [Hug22], the third author proved the conjecture for quadratic rational maps. Ji and Xie later proved the general case:

Theorem 1 ([JX23, Theorem 1.12])**.** *Assume that $K \subset \mathbb{C}$ is an imaginary quadratic field and $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ whose multipliers all lie in $\mathcal{O}_K$. Then f is a power map, a Chebyshev map or a Lattès map.*

In this note, we present a variant of Ji and Xie's proof of Theorem 1. Our proof only differs from the original one in one of the arguments: where they use a result from non-Archimedean dynamics proved by Rivera-Letelier in [RL03], we consider a differential equation instead. Thus, our main contribution is Proposition 13.

We also determine the entire maps with integer multipliers. A nonaffine entire map $f\colon \mathbb{C} \to \mathbb{C}$ is said to be a *power map* or a *Chebyshev map* if it is polynomial and it is a power map or a Chebyshev map in the previous sense. Equivalently, a nonaffine entire map $f\colon \mathbb{C} \to \mathbb{C}$ is

- a power map if it is conjugate to $z \mapsto z^d$ for some integer $d \geq 2$,
- a Chebyshev map if it is conjugate to $\pm T_d$ for some integer $d \geq 2$.

Also note that Lattès maps are not polynomial.

Our arguments to prove Theorem 1 also apply to the case of entire maps. Thus, we obtain the result below, which shows that there is no transcendental entire map whose multipliers all lie in the ring of integers of some imaginary quadratic field.

Theorem 2. *Assume that $K \subset \mathbb{C}$ is an imaginary quadratic field and $f\colon \mathbb{C} \to \mathbb{C}$ is a nonaffine entire map whose multipliers all lie in $\mathcal{O}_K$. Then f is a power map or a Chebyshev map.*

Remark 3. In fact, our proof shows that the conclusions of Theorems 1 and 2 still hold if one only assumes that there is some open set U that intersects the Julia set $\mathcal{J}_f$ of f and such that the multipliers of f at its periodic points in U all lie in $\mathcal{O}_K$. This is also true of Ji and Xie's proof of Theorem 1.

After writing this note, the third author obtained the following stronger version of Theorem 1:

Theorem 4 ([Hug23, Main Theorem])**.** *Assume that $K \subset \mathbb{C}$ is a number field and $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ whose multipliers all lie in K. Then f is a power map, a Chebyshev map or a Lattès map.*

In contrast, the first and third authors together with Gorbovickis later showed that Theorem 2 does not generalize to the case of rational multipliers:

Theorem 5 ([BGH23, Theorem 4])**.** *Assume that $K \subset \mathbb{C}$ is a number field that is not contained in $\mathbb{R}$. Then there exist transcendental entire maps $f\colon \mathbb{C} \to \mathbb{C}$ whose multipliers all lie in K.*

In Section 2, we present a characterization of power maps, Chebyshev maps and Lattès maps. In Section 3, we present our proof of Theorems 1 and 2.

2. Exceptional maps and escaping quadratic-like maps

Throughout this section, we assume that S denotes either $\mathbb{C}$ or $\widehat{\mathbb{C}}$. We say that a holomorphic map $f\colon S \to S$ is *nonlinear* if it is neither constant nor injective. In other words, the nonlinear holomorphic maps $f\colon S \to S$ are precisely the nonaffine entire maps if $S = \mathbb{C}$ and the rational maps of degree $d \geq 2$ if $S = \widehat{\mathbb{C}}$.

2.1. **Exceptional maps.** We say that a nonlinear holomorphic map $f\colon S \to S$ is *exceptional* if it is a power map, a Chebyshev map or a Lattès map.

Ritt obtained the following characterization of exceptional maps:

Lemma 6 ([Rit22]). *Suppose that $f\colon S \to S$ is a nonlinear holomorphic map, $\phi\colon \mathbb{C} \to S$ is a nonconstant holomorphic map, $\alpha\colon \mathbb{C} \to \mathbb{C}$ is an affine map that is not a translation and $\tau\colon \mathbb{C} \to \mathbb{C}$ is a nontrivial translation such that*

$$\phi \circ \alpha = f \circ \phi \quad \text{and} \quad \phi \circ \tau = \phi\,.$$

Then f is exceptional.

Remark 7. In fact, Ritt is interested in the equation $\phi \circ \alpha = f \circ \phi$, with $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a rational map, $\alpha\colon \mathbb{C} \to \mathbb{C}$ a nontrivial homothety about the origin and $\phi\colon \mathbb{C} \to \widehat{\mathbb{C}}$ a periodic and nonconstant holomorphic map, and he seeks to find ϕ. As he points out in the last sentence of his introduction, his arguments also apply if $f\colon \mathbb{C} \to \mathbb{C}$ is an entire map and $\phi\colon \mathbb{C} \to \mathbb{C}$ is a periodic and nonconstant entire map. We may also take $\alpha\colon \mathbb{C} \to \mathbb{C}$ to be any affine map that is not a translation, conjugating α if necessary to reduce to the previous situation. Finally, one easily deduces from the form of ϕ found by Ritt that f is necessarily exceptional if it is nonlinear.

The following generalization of Lemma 6 is essentially due to Ji and Xie (compare [JX23, Lemma 2.9]).

Lemma 8. *Suppose that $f\colon S \to S$ is a nonlinear holomorphic map, $\phi\colon \mathbb{C} \to S$ is a nonconstant holomorphic map and $\alpha_1\colon \mathbb{C} \to \mathbb{C}$ and $\alpha_2\colon \mathbb{C} \to \mathbb{C}$ are affine maps that do not commute and such that*

$$\phi \circ \alpha_1 = f \circ \phi = \phi \circ \alpha_2\,.$$

Then f is exceptional.

Proof. Note that α_1 or α_2 is not a translation as, otherwise, they would commute. Also note that α_1 and α_2 are both nonconstant because f and ϕ are not constant. Define the affine map

$$\tau = \alpha_1 \circ \alpha_2^{-1} \circ \alpha_1 \circ \alpha_2 \circ \left(\alpha_1^{-1}\right)^{\circ 2}\,.$$

Then τ is a translation as the linear endomorphism associated with a composition of affine maps equals the composition of the associated linear endomorphisms and linear endomorphisms of $\mathbb{C}$ commute. Also note that τ is not the identity map as, otherwise, α_1 and α_2 would commute. Thus, $\tau\colon \mathbb{C} \to \mathbb{C}$ is a nontrivial translation. Moreover, we have

$$\phi \circ \alpha_1 \circ \alpha_2 = f \circ \phi \circ \alpha_2 = f^{\circ 2} \circ \phi = f \circ \phi \circ \alpha_1 = \phi \circ \alpha_1^{\circ 2}\,,$$

and hence

$$\phi \circ \tau = \phi \circ \alpha_1 \circ \alpha_2^{-1} \circ \alpha_1 \circ \alpha_2 \circ \left(\alpha_1^{-1}\right)^{\circ 2} = \phi \circ \alpha_1 \circ \alpha_2 \circ \left(\alpha_1^{-1}\right)^{\circ 2} = \phi\,.$$

Therefore, f is exceptional by Lemma 6, and the lemma is proved. ∎

2.2. **Escaping quadratic-like maps.** An *escaping quadratic-like map* is a holomorphic covering map $f\colon U \to V$ of degree 2, with U, V nonempty open subsets of $\mathbb{C}$ such that $U \Subset V$ and V is simply connected. In this situation, note that U has two connected components U_1 and U_2, which are mapped biholomorphically onto V by f.

Remark 9. The notion of escaping quadratic-like map is related to the well-known one of quadratic-like map as follows: Recall that a *quadratic-like map* is a proper holomorphic map $f\colon V \to W$ of degree 2, with $V \Subset W$ nonempty simply connected open subsets of $\mathbb{C}$. In this situation, $f\colon V \to W$ has a unique critical point $\gamma \in V$. If $f(\gamma) \in W \setminus V$, then $f\colon f^{-1}(V) \to f^{-1}(W)$ is an escaping quadratic-like map.

We shall use the result below, which was proved by Bergweiler. His proof relies on a weak version of the Ahlfors five islands theorem. We give here a proof in the case of rational maps, which follows Ji and Xie's proof of Theorem 1. We refer the reader to Bergweiler's article for a proof of the general case.

Lemma 10 ([Ber00, Proposition B.3])**.** *Suppose that $f\colon S \to S$ is a nonlinear holomorphic map. Then there exist an integer $n \geq 1$ and open subsets U, V of $\mathbb{C}$ such that $f^{\circ n}\colon U \to V$ is an escaping quadratic-like map.*

Proof in the case of rational maps. Suppose here that $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$. Then f has infinitely many repelling periodic points. Moreover, f has only finitely many critical points and, hence, also only finitely many periodic points that lie in the forward orbit of a critical point. Therefore, f has a repelling periodic point $z_1 \in \mathbb{C}$, with period $p \geq 1$, that does not lie in the forward orbit of a critical point. There exist a simply connected open neighborhood V_1 of z_1 in $\mathbb{C}$ and a local inverse $g_1\colon V_1 \to g_1(V_1)$ of $f^{\circ p}$ such that $g_1(z_1) = z_1$ and $g_1(V_1) \Subset V_1$. Now, z_1 lies in the Julia set $\mathcal{J}_f$ of f and its iterated preimages accumulate on all of $\mathcal{J}_f$, and hence there exist $\ell \geq 1$ and $z_2 \in V_1 \setminus \{z_1\}$ such that $f^{\circ \ell}(z_2) = z_1$. The point z_2 is not critical for $f^{\circ \ell}$ by the definition of z_1, and hence there exist a simply connected open neighborhood $V \subset V_1$ of z_1 and a local inverse $g_2\colon V \to g_2(V)$ of $f^{\circ \ell}$ such that $g_2(z_1) = z_2$ and $g_2(V) \Subset V_1 \setminus \{z_1\}$. Now, note that $g_1\colon V_1 \to g_1(V_1)$ is a contracting map with respect to the Poincaré metric on V_1 since $g_1(V_1) \Subset V_1$. Therefore, as $g_1(z_1) = z_1$, $g_2(V) \Subset V_1 \setminus \{z_1\}$ and $V \subset V_1$, there exist $m_1, m_2 \geq 1$ such that

$$W_2 = g_1^{\circ m_1} \circ g_2(V) \Subset V \quad \text{and} \quad W_1 = g_1^{\circ m_2}(V) \Subset V \setminus W_2\,.$$

Define

$$h_1 = g_1^{\circ m_2}\colon V \to W_1 \quad \text{and} \quad h_2 = g_1^{\circ m_1} \circ g_2\colon V \to W_2\,.$$

Note that h_1 is a local inverse of $f^{\circ n_1}$, with $n_1 = m_2 p$, and h_2 is a local inverse of $f^{\circ n_2}$, with $n_2 = m_1 p + \ell$. Set

$$n = n_1 n_2 \quad \text{and} \quad U = h_1^{\circ n_2}(V) \cup h_2^{\circ n_1}(V)\,.$$

Then $f^{\circ n}\colon U \to V$ is an escaping quadratic-like map. This completes the proof of the lemma in the case of rational maps. □

2.3. **Affine escaping quadratic-like maps.** We say that an escaping quadratic-like map $f\colon U \to V$ is *affine* if it is affine on each of the two connected components of U. We say that two escaping quadratic-like maps $f_1\colon U_1 \to V_1$ and $f_2\colon U_2 \to V_2$ are *conjugate* if there exists a biholomorphism $\phi\colon V_1 \to V_2$ such that $\phi \circ f_1 = f_2 \circ \phi$ on U_1.

In our proof of Theorems 1 and 2, we shall use the following characterization of exceptional maps:

Lemma 11. *Suppose that $f\colon S \to S$ is a nonlinear holomorphic map such that an escaping quadratic-like map of the form $f^{\circ n}\colon U \to V$, with $n \geq 1$ and $U, V \subset \mathbb{C}$, is conjugate to an affine escaping quadratic-like map. Then f is exceptional.*

Proof. By hypothesis, there exist an affine escaping quadratic-like map $g\colon U' \to V'$ and a biholomorphism $\phi\colon V' \to V$ such that $\phi \circ g = f^{\circ n} \circ \phi$ on U'. Denote by U_1' and U_2' the two connected components of U'. Then the restrictions of g to U_1' and U_2' agree with the restrictions of affine maps $\alpha_1 \colon \mathbb{C} \to \mathbb{C}$ and $\alpha_2 \colon \mathbb{C} \to \mathbb{C}$. We have $\phi \circ \alpha_1 = f^{\circ n} \circ \phi$ on U_1' and $\phi \circ \alpha_2 = f^{\circ n} \circ \phi$ on U_2'. As the affine maps α_1 and α_2 are repelling, we may use any of these two relations to extend ϕ to a holomorphic map $\widehat{\phi}\colon \mathbb{C} \to S$. By the identity principle, we have

$$\widehat{\phi} \circ \alpha_1 = f^{\circ n} \circ \widehat{\phi} = \widehat{\phi} \circ \alpha_2 \,.$$

Furthermore, α_1 and α_2 have distinct fixed points, in U_1' and U_2' respectively, and hence they do not commute. Therefore, the map $f^{\circ n}$ is exceptional by Lemma 8, and hence so is f. Thus, the lemma is proved. □

Remark 12. The proof above uses the fact that, if $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ such that $f^{\circ n}$ is exceptional for some $n \geq 1$, then so is f. This can be deduced from the following facts. Any postcritically finite rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$ has an associated orbifold O_f. A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$ is an exceptional map if and only if it is postcritically finite and its orbifold O_f is parabolic. If $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ and $n \geq 1$, then f is postcritically finite if and only if $f^{\circ n}$ is postcritically finite and, in this case, we have $O_f = O_{f^{\circ n}}$. We refer the reader to [BM17] for further details.

3. Proof of the results

As for entire and rational maps, we can define the notions of periodic point and multiplier for escaping quadratic-like maps. It follows from Lemmas 10 and 11 that Theorems 1 and 2 are a consequence of the following result:

Proposition 13. *Assume that K is an imaginary quadratic field and $f\colon U \to V$ is an escaping quadratic-like map whose multipliers all lie in $\mathcal{O}_K$. Then f is conjugate to an affine escaping quadratic-like map.*

Our proof of this statement will occupy the rest of the note. Assume from now on that $f\colon U \to V$ is an escaping quadratic-like map whose multipliers all lie in the ring of integers $\mathcal{O}_K$ of some imaginary quadratic field K. Denote by U_1 and U_2 the two connected components of U and define

$$f_1 = f|_{U_1} \,, \quad f_2 = f|_{U_2} \,, \quad g_1 = f_1^{-1} \colon V \to U_1 \,, \quad g_2 = f_2^{-1} \colon V \to U_2 \,.$$

As $U \Subset V$, the maps g_1 and g_2 are contracting with respect to the Poincaré metric on V. Therefore, the maps f_1 and f_2 have unique fixed points $z_1 \in U_1$ and $z_2 \in U_2$ respectively, which are repelling. Denote by λ_1 and λ_2 their associated multipliers, which lie in $\mathcal{O}_K$.

By the Koenigs linearization theorem, the sequence $(\phi_n)_{n \geq 0}$ of univalent maps defined on V by

$$\phi_n(z) = \lambda_1^n \left(g_1^{\circ n}(z) - z_1\right)$$

converges to a univalent map $\phi\colon V \to \mathbb{C}$ such that $\phi \circ f_1 = \lambda_1 \phi$. Thus, replacing f by $\phi \circ f \circ \phi^{-1}$ if necessary, we may assume that $f_1(z) = \lambda_1 z$ for all $z \in U_1$, which yields $U_1 = \frac{1}{\lambda_1} V$ and $z_1 = 0$. We shall prove that f_2 is affine.

3.1. **A special sequence of periodic points.** Still following Ji and Xie's proof, we consider a particular sequence of periodic points for f. For each $n \geq 1$, the map $g_2 \circ g_1^{\circ(n-1)}\colon V \to U_2$ is contracting with respect to the Poincaré metric on V, and hence it has a unique fixed point $w_n \in U_2$. For every $n \geq 1$, we have

$$\forall j \in \{1, \ldots, n\},\ f^{\circ j}(w_n) = g_1^{\circ(n-j)}(w_n) = \frac{w_n}{\lambda_1^{n-j}}\,.$$

In particular, for every $n \geq 1$, the point w_n is periodic for f with period n and its associated multiplier ρ_n satisfies

$$\rho_n = \prod_{j=1}^{n} f'\left(\frac{w_n}{\lambda_1^{n-j}}\right) = \lambda_1^{n-1} f_2'(w_n) \in \mathcal{O}_K\,.$$

Note that

$$w_n = g_2\left(\frac{w_n}{\lambda_1^{n-1}}\right) = \alpha + \frac{\beta}{\lambda_1^{n-1}} + o\left(\frac{1}{\lambda_1^n}\right) \quad \text{as} \quad n \to +\infty\,, \quad \text{with} \quad \begin{cases} \alpha = g_2(0) \\ \beta = \alpha g_2'(0) \end{cases},$$

since $\lim_{n\to+\infty} w_n = \alpha$, and hence

$$\rho_n = \lambda_1^{n-1} f_2'(w_n) = a\lambda_1^{n-1} + b + o(1) \quad \text{as} \quad n \to +\infty\,, \quad \text{with} \quad \begin{cases} a = f_2'(\alpha) \\ b = \beta f_2''(\alpha) \end{cases}.$$

Now, let us use the assumption that the multipliers of f all lie in $\mathcal{O}_K$ to obtain the following:

Claim 14. We have $\rho_n = a\lambda_1^{n-1} + b$ for all n sufficiently large.

Proof. Write $\rho_n = a\lambda_1^{n-1} + b + \varepsilon_n$ for $n \geq 1$, so that $\lim_{n\to+\infty} \varepsilon_n = 0$. Then, for every $n \geq 1$, we have

$$\lambda_1 \rho_n - \rho_{n+1} = (\lambda_1 - 1)\, b + \lambda_1 \varepsilon_n - \varepsilon_{n+1} \in \mathcal{O}_K\,.$$

Moreover, $\lim_{n\to+\infty} (\lambda_1 \varepsilon_n - \varepsilon_{n+1}) = 0$. It follows that $(\lambda_1 - 1)\, b \in \mathcal{O}_K$ because $\mathcal{O}_K$ is closed in $\mathbb{C}$. Therefore, since $\mathcal{O}_K$ is discrete, for every n sufficiently large, we have $\lambda_1 \rho_n - \rho_{n+1} = (\lambda_1 - 1)\, b$, and hence $\varepsilon_{n+1} = \lambda_1 \varepsilon_n$. As $|\lambda_1| > 1$ and $\lim_{n\to+\infty} \varepsilon_n = 0$, this yields $\varepsilon_n = 0$ for all n sufficiently large. Thus, the claim is proved. ☐

3.2. **A differential equation.** Our proof now deviates from Ji and Xie's proof of Theorem 1. Instead of using a result concerning the non-Archimedean dynamics of rational maps, we show that f_2 is solution of a simple differential equation.

Claim 15. The holomorphic map $f_2\colon U_2 \to V$ satisfies

$$\text{(E)} \qquad \forall z \in U_2,\ f_2'(z) = a + b\frac{f_2(z)}{z}\,.$$

Proof. For every n sufficiently large, we have

$$f_2'(w_n) = \frac{\rho_n}{\lambda_1^{n-1}} = a + \frac{b}{\lambda_1^{n-1}} = a + b\frac{f_2(w_n)}{w_n}\,.$$

Therefore, since $(w_n)_{n\geq1}$ accumulates at $\alpha \in U_2$, the relation (E) follows from the identity principle. Thus, the claim is proved. ☐

Note that, as (E) is a first-order linear ordinary differential equation, it may be easily solved. However, we shall not use the explicit form of the solutions.

Claim 16. We have

$$\frac{f_2'(\alpha)}{\lambda_2} = 1 + \frac{\nu}{1-\lambda_2}\,, \quad \text{with} \quad \alpha = g_2(0) \quad \text{and} \quad \nu = z_2 \frac{f_2''(z_2)}{f_2'(z_2)}\,.$$

Proof. Evaluating (E) at $z = z_2$, we obtain

$$\lambda_2 = a + b = f_2'(\alpha) + b\,.$$

Moreover, differentiating (E) and evaluating at $z = z_2$, we obtain

$$f_2''(z_2) = \frac{b}{z_2}(\lambda_2 - 1)\,.$$

Therefore, we have

$$\nu = z_2 \frac{f_2''(z_2)}{\lambda_2} = \frac{b}{\lambda_2}(\lambda_2 - 1) = \left(1 - \frac{f_2'(\alpha)}{\lambda_2}\right)(\lambda_2 - 1)\,,$$

which may be rewritten in the desired form. Thus, the claim is proved. □

3.3. **Conclusion.** For $k \geq 1$, consider the map $f^{[k]} \colon U_1 \cup g_2^{\circ k}(V) \to V$ defined by

$$f^{[k]}(z) = \begin{cases} \lambda_1 z & \text{if } z \in U_1 \\ f_2^{\circ k}(z) & \text{if } z \in g_2^{\circ k}(V) \end{cases}.$$

For every $k \geq 1$, the map $f^{[k]}$ is an escaping quadratic-like map whose multipliers all lie in $\mathcal{O}_K$. Moreover, for every $k \geq 1$, the map $f^{[k]}$ fixes z_2 with multiplier λ_2^k. Therefore, for every $k \geq 1$, applying Claim 16 with $f^{[k]}$ instead of f, we obtain

$$\frac{\left(f_2^{\circ k}\right)'(\alpha_k)}{\lambda_2^k} = 1 + \frac{\nu_k}{1-\lambda_2^k}\,, \quad \text{with} \quad \alpha_k = g_2^{\circ k}(0) \quad \text{and} \quad \nu_k = z_2 \frac{\left(f_2^{\circ k}\right)''(z_2)}{\left(f_2^{\circ k}\right)'(z_2)}\,.$$

Now, an elementary calculation – which is the composition rule for nonlinearities – shows that

$$\nu_k = \sum_{j=0}^{k-1} \lambda_2^j \nu_1 = \frac{1-\lambda_2^k}{1-\lambda_2}\nu_1$$

for all $k \geq 1$, and in particular

$$\frac{\nu_k}{1-\lambda_2^k} = \frac{\nu_1}{1-\lambda_2}$$

does not depend on $k \geq 1$. Therefore, for every $k \geq 1$, we have

$$\frac{f_2'(\alpha_{k+1})}{\lambda_2} \cdot \frac{\left(f_2^{\circ k}\right)'(\alpha_k)}{\lambda_2^k} = \frac{\left(f_2^{\circ(k+1)}\right)'(\alpha_{k+1})}{\lambda_2^{k+1}} = \frac{\left(f_2^{\circ k}\right)'(\alpha_k)}{\lambda_2^k}\,,$$

and hence $f_2'(\alpha_{k+1}) = \lambda_2$. As $(\alpha_k)_{k\geq 2}$ accumulates at $z_2 \in U_2$, it follows from the identity principle that

$$\forall z \in U_2,\ f_2'(z) = \lambda_2\,.$$

Thus, the map f_2 is affine, which completes the proof of Proposition 13.

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